Int. J. Solids Structures, 1966, Vol. 2, pp. 361 to 369. Pergamon Press Ltd. Printed in Great Britain

ON THE STABILITY OF A STRUT UNDER UNIFORMLY DISTRIBUTED AXIAL FORCES

R. FRISCH-FAY

Department of Structures, School of Civil Engineering, The University of New South Wales, Sydney, Australia

Abstract—Eigenvalue problems governed by a differential equation with variable coefficients may sometimes be reduced to a lower order equation by an appropriate change in the variables. If this transformation results in a non-homogeneous equation the problem of finding a particular integral will arise which is likely to present difficulties if the complementary function contains Bessel functions. By introducing two Bessel integrals, $A_1(z)$ and $B_1(z)$ and one multiple Bessel integral $D_1(z)$, the critical distributed load of a uniform strut, prevented from sway at top and bottom, can be found analytically. Functions A_1, B_1, and D_1 are worked and tabulated.

INTRODUCTION

FOR THE STRUT shown in Fig. 1 the fundamental relationship $EIy''' = -S$ can immediately be applied at a point $C(w, y)$. It becomes

$$EI\,\mathrm{d}^3y/\mathrm{d}w^3 = -(l-w)q\sin\theta+H \tag{1}$$

The slope being small we substitute $\sin\theta$ by $\theta \approx \tan\theta = \mathrm{d}y/\mathrm{d}w = u(w)$ and reduce equation (1)

$$EI\,\mathrm{d}^2u/\mathrm{d}w^2+(l-w)qu = H \tag{2}$$

Introducing $l-w = x$, $m^2 = q/EI$, and $p = H/EI$ the differential equation obtains its final form

$$\mathrm{d}^2u/\mathrm{d}x^2+m^2xu = p \tag{3}$$

If the strut is fixed at the bottom, completely free at the top and subject to the same loading as the strut in Fig. 1 analysis follows the same steps as before up to equation (2) but there is now no horizontal reaction entering the right hand side. Thus equation (2) becomes

$$EI\,\mathrm{d}^2u/dx^2+(l-w)qu = 0 \tag{4}$$

and equation (3) changes to

$$\mathrm{d}^2u/\mathrm{d}x^2+m^2xu = 0 \tag{5}$$

The integral of equation (5) is

$$u_c = c_1\sqrt{x}\,J_{\frac{1}{3}}(z)+c_2\sqrt{x}\,J_{-\frac{1}{3}}(z) \tag{6}$$

where $z = \frac{2}{3}mx^{\frac{3}{2}}$. The critical load [1] for this case is $(ql)_{cr} = 7{\cdot}84EI/l^2$.

Numerical methods [2], [3] have been employed to find the critical load of a strut shown in Fig. 1 and it has been given as $(ql)_{cr} = 18{\cdot}87EI/l^2$.

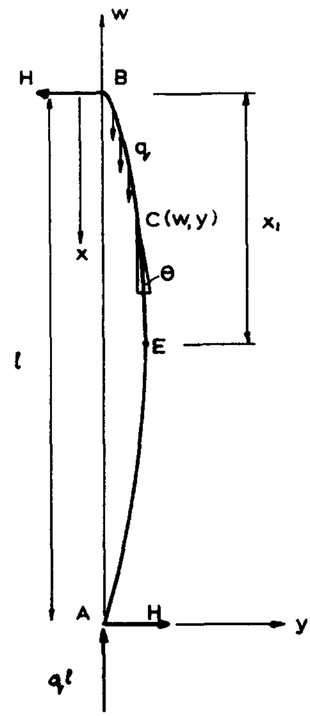

FIG. 1.

SOLUTION OF BASIC EQUATION

The most general method of finding a particular integral to (3), due to Lagrange, is to assign unknown functions v_1, and v_2 in place of c_1 and c_2, respectively, in equation (6), and then solve the equations

$$\left.\begin{aligned} v_1'\sqrt{x}\,J_{\frac{1}{3}}(z)+v_2'\sqrt{x}\,J_{-\frac{1}{3}}(z) &= 0 \\ v_1'[\sqrt{x}\,J_{\frac{1}{3}}(z)]'+v_2'[\sqrt{x}\,J_{-\frac{1}{3}}(z)]' &= p \end{aligned}\right\} \tag{7}$$

where the primes represent differentiation with respect to x. To find the derivatives of $J_{\frac{1}{3}}(z)$ and of $J_{-\frac{1}{3}}(z)$ the formulae

$$J_\nu'(z) = \left[J_{\nu-1}(z)-\frac{\nu}{z}J_\nu(z)\right]z'$$

and

$$J_\nu'(z) = \left[\frac{\nu}{z}J_\nu(z)-J_{\nu+1}(z)\right]z'$$

respectively, will be applied. These will transform equations (7) to

$$\left.\begin{aligned} v_1'\sqrt{x}\,J_{\frac{1}{3}}(z)+v_2'\sqrt{x}\,J_{-\frac{1}{3}}(z) &= 0 \\ v_1'mxJ_{-\frac{2}{3}}(z)-v_2'mxJ_{\frac{2}{3}}(z) &= p \end{aligned}\right\} \tag{8}$$

from which

and
$$\left.\begin{aligned} v_1' &= pJ_{-\frac{1}{3}}(z)/g(x) \\ v_2' &= -pJ_{\frac{1}{3}}(z)/g(x) \end{aligned}\right\} \tag{9}$$

where

$$g(x) = mx[J_{\frac{1}{3}}(z)J_{\frac{2}{3}}(z)+J_{-\frac{1}{3}}(z)J_{-\frac{2}{3}}(z)].$$

The function $g(x)$ can be simplified considerably by using the relationships

and
$$\left.\begin{aligned} J_{\frac{1}{3}}(z)J_{\frac{2}{3}}(z) &= \sum_{r=0}^{\infty} \frac{(-1)^r(1+2r)_r(\frac{1}{2}z)^{1+2r}}{r!\Gamma(r+\frac{4}{3})\Gamma(r+\frac{5}{3})} \\ J_{-\frac{1}{3}}(z)J_{-\frac{2}{3}}(z) &= \sum_{r=0}^{\infty} \frac{(-1)^r(1+2r)_r(\frac{1}{2}z)^{-1+2r}}{r!\Gamma(r+\frac{2}{3})\Gamma(r+\frac{1}{3})} \end{aligned}\right\} \tag{10) [4}$$

where

$$(a+2r)_r = (a+2r)(a+2r-1)(a+2r-2)\dots(a+r+1)$$

and

$$(a+2r)_0 = 1$$

Expanding equations (10) and adding the corresponding terms, all terms except the first of the second series and the last of the first series will cancel out. These remaining terms are

$$\frac{(\frac{1}{2}z)^{-1}}{\Gamma\frac{1}{3}\Gamma\frac{2}{3}}$$

and

$$\frac{(-1)^r(1+2r)!(\frac{1}{2}z)^{1+2r}}{r!(1+r)!(\frac{1}{3}+r)!(\frac{2}{3}+r)!}$$

with the latter vanishing in the limit as $r = \infty$

Therefore,

and
$$\left.\begin{aligned} v_1 &= p\tfrac{1}{3}\Gamma\tfrac{1}{3}\Gamma\tfrac{2}{3}\int\sqrt{x}\,J_{-\frac{1}{3}}(z)\,\mathrm{d}x+c_1 \\ v_2 &= -p\tfrac{1}{3}\Gamma\tfrac{1}{3}\Gamma\tfrac{2}{3}\int\sqrt{x}\,J_{\frac{1}{3}}(z)\,\mathrm{d}x+c_2 \end{aligned}\right\} \tag{11}$$

Combining (4), (9), and (11) the complete solution of (3) is

$$u = C_1B'(x)+C_2A'(x)+P[A'(x)B(x)-A(x)B'(x)] \tag{12}$$

where

$$A(x) = \gamma\int\sqrt{x}\,J_{-\frac{1}{3}}(z)\,\mathrm{d}x, \qquad B(x) = -\gamma\int\sqrt{x}\,J_{\frac{1}{3}}(z)\,\mathrm{d}x,$$
$$\gamma = \tfrac{1}{3}\Gamma\tfrac{1}{3}\Gamma\tfrac{2}{3} = 2\pi/3\sqrt{3}, \qquad C_1 = -c_1/\gamma, \qquad C_2 = c_2/\gamma,$$
$$P = p/\gamma.$$

PARTICULAR CASES

1. *Strut hinged at both ends*

In this particular case the bending moment is zero at $x = 0$ and $x = l$, or, expressed in another way, the boundary conditions are

$$u'(0) = 0 \quad \text{and} \quad u'(l) = 0 \tag{13}$$

From the first of these two conditions

$$C_1 B''(0) + C_2 A''(0) + P[A''(0)B(0) - A(0)B''(0)] = 0 \tag{14}$$

Since $A''(x) = -\gamma m x J_{\frac{1}{3}}(z)$, and $B''(x) = -\gamma m x J_{-\frac{1}{3}}(z)$ we have $B''(0) = -3\gamma m(m/3)^{-\frac{1}{3}}$; $A''(0) = 0$. Also, $A(0) = 0$. From this it follows that $C_1 = 0$.

The second boundary condition, $u'(l) = 0$, can be written as

$$C_2 A''(l) + P[A''(l)B(l) - A(l)B''(l)] = 0$$

whence

$$P/C_2 = \left.\frac{J_{\frac{1}{3}}(z)}{A(x)J_{-\frac{1}{3}}(z) - B(x)J_{\frac{1}{3}}(z)}\right|_{x=l} \tag{15}$$

As there can be no deflection at $x = 0$ we find from

$$\int u \, dx = C_2 A(x) + P\int [A'(x)B(x) - A(x)B'(x)] \, dx + C_3 = 0$$

that, since $A(0) = B(0) = B'(0) = 0$, C_3 is also zero.

The last boundary condition stipulates that there is no deflection at $x = l$. Hence

$$\int_0^l u \, dx = C_2 A(l) + P\int_0^l [A'(x)B(x) - A(x)B'(x)] \, dx = 0$$

or,

$$P/C_2 = \frac{A(l)}{\int_0^l [A(x)B'(x) - A'(x)B(x)] \, dx} \tag{16}$$

By equating (15) and (16) an equation in x is obtained the solution of which will yield the eigenvalue of (3) and hence the critical lȯad can be found. However, it will be convenient to introduce the functions

$$\left.\begin{aligned} A_1(z) &= \int_0^z J_{-\frac{1}{3}}(t) \, dt \\ B_1(z) &= \int_0^z J_{\frac{1}{3}}(t) \, dt \\ \text{and} \quad D_1(z) &= \tfrac{1}{2}\int_0^z [J_{-\frac{1}{3}}(t)B_1(t) - J_{\frac{1}{3}}(t)A_1(t)] \, dt \end{aligned}\right\} \tag{17}$$

With these new functions the equation that governs the critical load can be expressed as

$$\frac{J_{\frac{2}{3}}(z)}{J_{-\frac{2}{3}}(z)A_1(z)+J_{\frac{2}{3}}(z)B_1(z)}=\frac{\frac{1}{2}A_1(z)}{D_1(z)} \tag{18}$$

The functions $A_1(z)$, $B_1(z)$, and $D_1(z)$ do not appear to have been tabulated. Accordingly, they have been calculated by the writer and tabulated in the Appendix. The numerical solution of equation (18) will be given later.

2. *Strut fixed at the bottom and hinged at the top*

Because of the hinge at $x = 0$ conditions are similar to those outlined in equations (13) and (14), and as before $C_1 = 0$.

Since the slope vanishes at $x = l$, it follows that

$$P/C_2=\left.\frac{A'(x)}{A(x)B'(x)-A'(x)B(x)}\right|_{x=l} \tag{19}$$

The remaining boundary conditions are the same as in case 1. It follows that equations (16) and (19) will have to be equated in order to find the critical load. This leads to the solution of

$$\frac{J_{-\frac{1}{3}}(z)}{J_{-\frac{1}{3}}(z)B_1(z)-J_{\frac{1}{3}}(z)A_1(z)}=\frac{\frac{1}{2}A_1(z)}{D_1(z)} \tag{20}$$

for the calculation of the critical load. The numerical answer will follow below.

3. *Strut fixed at both ends*

It becomes obvious, upon applying the condition $u(0) = 0$, that $C_2 = 0$ in equation (12). Since $u(l) = 0$, we find

$$P/C_1=\left.\frac{B'(x)}{A(x)B'(x)-A'(x)B(x)}\right|_{x=l} \tag{21}$$

There remain the boundary conditions $\int u\,\mathrm{d}x = 0$ at $x = 0$ and $x = l$. Satisfaction of the first of these conditions reveals that the new integration constant vanishes, i.e. $C_3 = 0$; and from the second there follows

$$P/C_1=\frac{B(l)}{\int_0^l [A(x)B'(x)-A'(x)B(x)]\,\mathrm{d}x} \tag{22}$$

Equating (21) and (22), and changing from x to z one obtains

$$\frac{J_{\frac{1}{3}}(z)}{J_{-\frac{1}{3}}(z)B_1(z)-J_{\frac{1}{3}}(z)A_1(z)}=\frac{\frac{1}{2}B_1(z)}{D_1(z)} \tag{23}$$

for the solution of the critical load.

NUMERICAL RESULTS

Smallest non-trivial roots of equations (18), (20), and (23) are $z_1 = 2{\cdot}87$, $z_2 = 4{\cdot}83$, and $z_3 = 5{\cdot}76$, respectively. For case 1 we note that

$$\tfrac{2}{3}\sqrt{(q/EI)}\, l^{\frac{3}{2}} = 2{\cdot}87$$

from which $(ql)_{cr} = 18{\cdot}53\ EI/l^2$. The critical loads for the cases 2 and 3 are $52{\cdot}49\ EI/l^2$ and $74{\cdot}65\ EI/l^2$, respectively.

The position of the point E of the strut where the slope vanishes is of some interest. For case 1, for example, this point is calculated by finding that value of x at which $u = 0$. This leads to

$$u = C_2 A'(x) + P[A'(x)B(x) - A(x)B'(x)] = 0$$

or

$$P/C_2 = \frac{m}{\gamma}\frac{J_{-\frac{1}{3}}(z)}{J_{-\frac{1}{3}}(z)B_1(z) - J_{\frac{1}{3}}(z)A_1(z)} \tag{24}$$

Substituting $z = 2{\cdot}87$ in equation (15) will result in $P/C_2 = -0{\cdot}805\ m/\gamma$ and with this value on the left hand side of equation (24) the solution of this equation becomes $z = 1{\cdot}15$. But $z = \frac{2}{3}mx^{\frac{3}{2}}$ and $m = \sqrt{(18{\cdot}53/l^3)}$; it follows that point E is at a distance $x_1 = 0{\cdot}54l$ from the top of the strut.

For the cases 2 and 3 one obtains $x_2 = 0{\cdot}46l$ and $x_3 = 0{\cdot}54l$, respectively. These values are based on $P/C_2 = -0{\cdot}334m/\gamma$ (case 2), and $P/C_1 = 0{\cdot}393m/\gamma$ (case 3).

REFERENCES

[1] M. G. SALVADORI and R. J. SCHWARZ, *Differential Equations in Engineering Problems*. Prentice-Hall, New York (1954).

[2] H. L. RAJASEKHARIAH, Beam column under axially distributed load. UNICIV Report No. R7. The University of New South Wales, Sydney (1965).

[3] F. REINITZHUBER, Naeherungsformeln fuer des Knicken von Staeben mit linear veraenderlicher Laengskraft. *I.A.B.S.* **13**, 309 (1953).

[4] N. W. MCLACHLAN, *Bessel Functions for Engineers*. Oxford University Press (1934).

[5] G. N. WATSON, *A Treatise on the Theory of Bessel Functions*. Cambridge University Press (1945).

[6] K. KARAS, Tabellen fuer Besselsche Funktionen. *Z. angew. Math. Mech.* **16**, 248 (1936).

[7] P. MONTAGNE, *Tables Abrègèes de Puissances Entières*. Dunod (1958).

[8] BARLOW-COMRIE, *Tables*. E. & F. N. Spon (1946).

APPENDIX

Term by term integration of $J_{-\frac{1}{3}}(z)$ resulted in the following series for the computation of $A_1(z)$,

$$A_1(z) = \frac{2}{\Gamma\frac{2}{3}}(\tfrac{1}{2}z)^{\frac{2}{3}}\left\{\frac{1}{\frac{2}{3}} - \frac{(\frac{1}{2}z)^2}{\frac{2}{3}\,.\,2\frac{2}{3}} + \frac{(\frac{1}{2}z)^4}{2!\frac{2}{3}\,.\,1\frac{2}{3}\,.\,4\frac{2}{3}} - \frac{(\frac{1}{2}z)^6}{3!\frac{2}{3}\,.\,1\frac{2}{3}\,.\,2\frac{2}{3}\,.\,6\frac{2}{3}} + \cdots\right.$$

Similarly, the computation of $B_1(z)$ was based on the series

$$\frac{2}{\Gamma 1\frac{1}{3}}(\tfrac{1}{2}z)^{\frac{1}{3}}\left\{\frac{1}{1\frac{1}{3}}-\frac{(\frac{1}{2}z)^2}{1\frac{1}{3}\,.\,3\frac{1}{3}}+\frac{(\frac{1}{2}z)^4}{2!1\frac{1}{3}\,.\,2\frac{1}{3}\,.\,5\frac{1}{3}}-\frac{(\frac{1}{2}z)^6}{3!1\frac{1}{3}\,.\,2\frac{1}{3}\,.\,3\frac{1}{3}\,.\,7\frac{1}{3}}+\cdots\right.$$

Tables of $A_1(z)$, $B_1(z)$, and $D_1(z)$

z	$A_1(z)$	$B_1(z)$	$-D_1(z)$	z	$A_1(z)$	$B_1(z)$	$-D_1(z)$
0·0	0·00000	0·00000	0·00000	5·0	0·65241	0·84980	0·96906
0·1	0·30040	0·03092	0·00155	5·1	0·65461	0·82027	0·95853
0·2	0·47552	0·07773	0·00619	5·2	0·66029	0·79311	0·94735
0·3	0·62019	0·13298	0·01388	5·3	0·66932	0·76855	0·93572
0·4	0·74639	0·19412	0·02459	5·4	0·68155	0·74678	0·92380
0·5	0·85882	0·25963	0·03822	5·5	0·69680	0·72796	0·91178
0·6	0·95984	0·32835	0·05470	5·6	0·71486	0·71223	0·89985
0·7	1·05080	0·39935	0·07391	5·7	0·73551	0·69969	0·88816
0·8	1·13253	9·47181	0·09573	5·8	0·75849	0·69040	0·87688
0·9	1·20556	0·54500	0·12000	5.9	0·78354	0·68440	0·86619
1·0	1·27030	0·61826	0·14657	6·0	0·81038	0·68169	0·85626
1·1	1·32703	0·69098	0·17526	6·1	0·83871	0·68225	0·84721
1·2	1·37599	0·76257	0·20588	6·2	0·86823	0·68601	0·83920
1·3	1·41742	0·83250	0·23824	6·3	0·89862	0·69289	0·83237
1·4	1·45151	0·90028	0·27212	6·4	0·92958	0·70277	
1·5	1·47848	0·96544	0·30730	6·5	0·96078	0·71551	
1·6	1·49856	1·02757	0·34356	6·6	0·99192	0·73093	
1·7	1·51201	1·08628	0·38067	6·7	1·02269	0·74884	
1·8	1·51907	1·14122	0·41839	6·8	1·05278	0·76903	
1·9	1·52005	1·19210	0·45651	6·9	1·08192	0·79126	
2·0	1·51526	1·23864	0·49476	7·0	1·10982	0·81529	
2·1	1·50502	1·28062	0·53293	7·1	1·13624	0·84085	
2·2	1·48969	1·31786	0·57079	7·2	1·16092	0·86767	
2·3	1·46967	1·35022	0·60811	7·3	1·18364	0·89545	
2·4	1·44532	1·37761	0·64469	7·4	1·20421	0·92393	
2·5	1·41708	1·39996	0·68032	7·5	1·22246	0·95279	
2·6	1·38537	1·41728	0·71480	7·6	1·23823	0·98176	
2·7	1·35063	1·42957	0·74795	7·7	1·25141	1·01055	
2·8	1·31331	1·43691	0·77960	7·8	1·26189	1·03886	
2·9	1·27385	1·43941	0·80960	7·9	1·26961	1·06644	
3·0	1·23272	1·43721	0·83782	8·0	1·27452	1·09300	
3·1	1·19037	1·43049	0·86412	8·1	1·27661	1·11832	
3·2	1·14725	1·41945	0·88841	8·2	1·27589	1·14212	
3·3	1·10380	1·40434	0·91060	8·3	1·27241	1·16423	
3·4	1·06046	1·38543	0·93061	8·4	1·26623	1·18443	
3·5	1·01765	1·36302	0·94839	8·5	1·25744	1·20253	
3·6	0·97577	1·33743	0·96393	8·6	1·24618	1·21840	
3·7	0·93520	1·30899	0·97720	8·7	1·23256	1·23189	
3·8	0·89631	1·27806	0·98821	8·8	1·21676	1·24290	
3·9	0·85942	1·24501	0·99697	8·9	1·19897	1·25136	
4·0	0·82486	1·21021	1·00354	9·0	1·17937	1·25719	
4·1	0·79289	1·17405	1·00798	9·1	1·15818	1·26038	
4·2	0·76378	1·13692	1·01037	9·2	1·13563	1·26092	
4·3	0·73773	1·09920	1·01078	9·3	1·11196	1·25884	
4·4	0·71494	1·06129	1·00934	9·4	1·08742	1·25418	
4·5	0·69555	1·02354	1·00614	9·5	1·06224	1·24702	
4·6	0·67967	0·98635	1·00134	9·6	1·03670	1·23745	
4·7	0·66739	0·95004	0·99509	9·7	1·01108	1·22561	
4·8	0·65876	0·91498	0·98751	9·8	0·98558	1·21162	
4·9	0·65377	0·88146	0·97878	9·9	0·96049	1·19565	
				10·0	0·93604	1·17787	

The numerical treatment of $D_1(z)$ requires a few preliminary modifications. From (17) we have

$$D_1(z) = \int_0^z \left[J_{-\frac{1}{3}}(\varphi) \sum_{r=0}^{\infty} J_{\frac{2}{3}+2r}(\varphi) - J_{\frac{1}{3}}(\varphi) \sum_{r=0}^{\infty} J_{\frac{4}{3}+2r}(\varphi) \right] d\varphi \tag{25}$$

On multiplying the appropriate Bessel functions in the summations indicated and collecting the terms in ascending powers the integrand can be expressed as

$$\sum_{r=0}^{\infty} \alpha_{2r+1} (\tfrac{1}{2}\varphi)^{2r+1} \tag{26}$$

where

$$\alpha_{2r+1} = \sum_{k=0}^{r} {}_{2r+1}C_k (-1)^k \left[\frac{1}{\Gamma(\frac{2}{3}+k)\Gamma[s-(\frac{2}{3}+k)]} - \frac{1}{\Gamma(\frac{4}{3}+k)\Gamma[s-(\frac{4}{3}+k)]} \right]$$

$$s = 2r+3$$

and ${}_{2r+1}C_k = (2r+1)!/k!(2r+1-k)!$ = number of combinations of $2r+1$ things taken k at a time. The term by term integral of (26) finally results in the series

$$D_1(z) = \sum_{r=0}^{\infty} \frac{\alpha_{2r+1}}{r+1} (\tfrac{1}{2}z)^{2r+2} \tag{27}$$

Bessel functions of order $+\frac{1}{3}$ and $-\frac{1}{3}$ needed for the solution of the critical loads and for the position of the points E are interpolated values from Watson's tables [5], while $J_{-\frac{2}{3}}$ and $J_{\frac{2}{3}}$ are given by Karas [6]. The actual computation of the series $A_1(z)$, $B_1(z)$, and $D_1(z)$ was carried out in matrix form on the UTECOM computer of the University of New South Wales. Terms up to 36th power have been included in the slowly converging series to ensure five decimal accuracy. Values of integer powers were taken from Montagne's tables [7] and values of $z^{\frac{2}{3}}$ and $z^{\frac{4}{3}}$ from Barlow-Comrie [8].

(Received 7 April 1965 and in revised form 28 October 1965)

Résumé—Les problèmes de valeurs propres gouvernés par une équation differentielle avec des coefficients variables peut quelquefois être réduits en une équation d'ordre inférieur par un changement approprié dans les variables. Si cette transformation a pour résultat une équation non-homogène le problème de trouver une intégrale particulière sers soulevé, qui pourrait présenter des difficultés si les fonctions complémentaires contiennent des fonctions Bessel. En introduisant deux intégrales Bessel, $A_1(z)$ et $B_1(z)$ et une intégrale multiple Bessel $D_1(z)$, la charge critique distribuée d'une colonne uniforme, empêché de mouvoir par le haut et le bas, peut être trouvé analytiquement. Les fonctions A_1, B_1 et D_1 sont calculées et cataloguées.

Zusammenfassung—Eigenwert Probleme, beschrieben durch eine Differentialgleichung mit veränderlichen Koeffizienten, können manchesmal zu einer niedrigeren Ordnung durch eine angemessene Veränderung in den Variablen, reduziert werden. Wenn diese Umwandlung eine nicht-homogene Gleichung ergibt, das Problem für die Ermittlung eines partikulären Integralen, wird sich erheben, welches wahrscheinlich Schwierigkeiten ergeben wird wenn die ergänzende Funktion, Bessel Funktionen enthält.

Durch Einführung von zwei Bessel Integralen, $A_1(z)$, und $B_1(z)$ und einem mehrfachen Bessel Integral $D_1(z)$, die kritische verteilte Belastung eines gleichförmigen Stabes, welche oben und unten gehaltern wird, kann analytisch gefunden werden. Die Funktionen A_1, B_1 und D_1 sind errechnet und in Tabellenform gegeben.

Абстракт—Задачи о собственных значениях, управляемые дифференциальным уравнением с переменными коэффициентами иногда могут быть сокращены до уравнения более низкого порядка соответствующим изменением переменных величин.

Если это преобразование в результате даёт неоднородное уравнение, то появляется проблема нахождения особого интеграла, что, возможно, будет представлять трудности, если дополнительная функция содержит функции Бесселя (Bessel).

При введении двух интегралов Бесселя, $A_1(z)$ и $B_1(z)$ и одного кратного интеграла Бесселя $D_1(z)$, критическая распределённая нагрузка однородной подпорки, предохранённой от иерельижений на верхушке и у основания, может быть найдена аналитически.

Функции A_1, B_1 и D_1, разработаны и табулированы (расположены в виде таблиц).